A Liouville theorem of degenerate elliptic equation and its application

Genggeng Huang*

(School of Mathematics Sciences, LMNS, Fudan University, Shanghai, China, 200433)

Abstract

In this paper, we apply the moving plane method to some degenerate elliptic equations to get a Liouville type theorem. As an application, we derive the a priori bounds for positive solutions of some semi-linear degenerate elliptic equations.

Key Words: Degenerate elliptic, moving plane, characteristic

1 Introduction

In the present paper, we study the nonnegative solutions $u(x, y)$ of the following equation with a constant $a > 1$

$$\begin{cases} yu_{yy} + au_y + \Delta_x u + u^\alpha = 0 \text{ in } R^{n+1}_+, n \geq 1, \\ u(x, y) \geq 0, u(x, y) \in C^2(\overline{R^{n+1}_+}), 1 < \alpha \leq \frac{n+2a+2}{n+2a-2}. \end{cases} \tag{1.1}$$

Notice that no boundary condition is imposed on $y = 0$ which is the characteristic of (1.1). As far as I know, I haven't seen any Liouville type theorem concerning (1.1). Set $x_{n+1} = 2\sqrt{y}$. (1.1) changes to

$$\Delta_{n+1}u + \frac{2a-1}{x_{n+1}}\partial_{n+1}u + u^\alpha = 0, \text{ in } R^{n+1}_+ \tag{1.2}$$

If $a = \frac{k}{2}, k \in N^+$, then we have

$$\begin{cases} \Delta_{x',\xi}v + v^\alpha = 0, \text{ in } R^{n+k}\backslash\{\xi_1 = 0, .., \xi_k = 0\} \\ v \geq 0, v \in C(R^{n+k}), 1 < \alpha \leq \frac{n+k+2}{n+k-2}. \end{cases} \tag{1.3}$$

with $v(x_1, ..., x_n, \xi_1, .., \xi_k) = u(x_1, ..., x_n, \sqrt{\xi_1^2 + ... + \xi_k^2})$. For (1.3), it is quite similar to the following problem except for a codimension hyperplane,

$$\begin{cases} \Delta u + u^q = \text{ in } R^n, n > 2, \\ u(x) \geq 0, \ u(x) \in C^2(R^n), 1 < q \leq \frac{n+2}{n-2}. \end{cases} \tag{1.4}$$

The above problem (1.4) was investigated in [11] and [2]. For the subcritical case $1 < q < \frac{n+2}{n-2}$, the only nonnegative solution $u(x)$ is trivial. Whereas for the critical case $q = \frac{n+2}{n-2}$, $u(x)$ is in a two parameter family of functions as

$$u_{t,x_0}(x) = \left(\frac{t\sqrt{n(n-2)}}{t^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}}. \tag{1.5}$$

*genggenghuang@fudan.edu.cn

There are many extended results of problem (1.4) which mainly concern on the higher order case,

$$\begin{cases}(-\Delta)^p u = u^q \text{ in } R^n, n > 2p,\\ u(x) \geq 0,\ u(x) \in C^{2p}(R^n), 1 < q \leq \frac{n+2p}{n-2p}.\end{cases} \tag{1.6}$$

For the case $p = 2$, Lin and Xu got the similar results respectively in [16] and [23]. Wei-Xu extended the results to the case $2 \leq 2p \leq n, p \in \mathbb{Z}$ in [22]. Chen-Li-Ou and Li proved the results for the most general case $0 < p < \frac{n}{2}$ by the integral form of the moving plane method(moving sphere method) respectively in [3] and [19]. Applying Chen-Li-Ou's method to systems as in [4] and [18], one can also get the similar conclusions. Chang and Yang in [6] also extended this results to manifolds.

The main method used in solving problem (1.4) is the moving plane method which was first proposed by Alexandrov [1] and developed by Serrin [21], Gidas, Ni and Nirenberg [9, 10]. Now moving plane method has been widely used in study of the symmetry of the positive solutions of many elliptic partial differential equations and systems. The key point of using the moving plane method in (1.4) is the conformal invariant property and the rotation invariant property of (1.4).

In our case, we also use the moving plane method and the conformal invariant property. To do so, we must establish some new maximal principles and overcome the difficult that (1.2) is not rotation invariant.

In this paper, we obtain the following results for (1.1).

Theorem 1.1. *Let $u(x, y) \geq 0$ be a solution to (1.1) with $a > 1$. Then*

(1) for $1 < \alpha < \frac{n+2a+2}{n+2a-2}$, $u(x, y) \equiv 0$,

(2) for $\alpha = \frac{n+2a+2}{n+2a-2}$, $$u_{t,x_0}(x, y) = \left(\frac{t\sqrt{(n+2a)(n+2a-2)}}{t^2 + 4y + |x - x_0|^2}\right)^{\frac{n+2a-2}{2}},$$

for some $x_0 \in R^n$ and $t \geq 0$.

It is easy to see that for $a = \frac{k}{2}, k \in N^+$, Theorem 1.1 is exactly the result of (1.4) in R^{n+k}. For general $a > 1$, we may consider Theorem 1.1 as the extension of the results of (1.4) to R^{n+2a} with real dimension $n + 2a$.

As an application of Theorem 1.1, we also derive a priori bounds for positive solutions of some semi-linear degenerate elliptic equations which arising from the study of geometry,

$$a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + f(x, u) = 0, \text{ in } \Omega \subset\subset R^2. \tag{1.7}$$

Let $\phi \in C^2(\mathcal{N}(\partial\Omega))$ be the defining function of $\partial\Omega$, namely,

$$\phi|_{\partial\Omega} = 0, \nabla\phi|_{\partial\Omega} \neq 0, \phi > 0 \text{ in } \Omega \cap \mathcal{N}(\partial\Omega) \tag{1.8}$$

where $\mathcal{N}(\partial\Omega)$ is a neighborhood of $\partial\Omega$. Also, suppose that

$$(a^{ij}) > 0 \text{ in } \Omega, a^{ij}(x)\partial_i\phi\partial_j\phi = 0, \nabla(a^{ij}\partial_i\phi\partial_j\phi) \neq 0 \text{ on } \partial\Omega \in C^2 \tag{1.9}$$

and that near $\partial\Omega$ for the eigenvalues of (a^{ij}) λ_1 and λ_2, there hold, for some constant c_0,

$$\lambda_1 \geq c_0 > 0, \lambda_2 = m(x)\phi, 0 < m(x) \in C(\bar{\Omega}). \tag{1.10}$$

Theorem 1.2. *Let (1.8), (1.9) and (1.10) be fulfilled. Suppose that $0 < u \in C^2(\Omega) \cap L^\infty(\Omega)$ solves (1.7) and that $a^{ij} \in C^2(\bar{\Omega}), b^i \in C^1(\bar{\Omega})$ and $f(x, t) \in C(\bar{\Omega} \times [0, \infty))$*

$$\lim_{t\to\infty} \frac{f(x, t)}{t^\alpha} = h(x),\ \textit{uniformly for some } 1 < \alpha < \frac{3 + 2a}{2a - 1}, \tag{1.11}$$

where $0 < h(x) \in C(\bar{\Omega})$,

$$a = \sup_{\partial\Omega} \frac{b^i \partial_i \phi - \partial_j a^{ij} \partial_i \phi}{\partial_k a^{ij} \partial_i \phi \partial_j \phi \phi^k} + 1, b \triangleq \inf_{\partial\Omega} \frac{b^i \partial_i \phi - \partial_j a^{ij} \partial_i \phi}{\partial_k a^{ij} \partial_i \phi \partial_j \phi \phi^k} > 1, \phi^k = \frac{\partial_k \phi}{|\nabla \phi|^2}. \tag{1.12}$$

Then it follows that

$$|u|_{L^\infty} \le C. \tag{1.13}$$

Remark 1.1. *Define*

$$g(x) = \frac{b^i \partial_i \phi - \partial_j a^{ij} \partial_i \phi}{\partial_k a^{ij} \partial_i \phi \partial_j \phi \phi^k} \text{ on } \partial\Omega \text{ where } a^{ij} \partial_i \phi \partial_j \phi = 0.$$

The invariance of $g(x)$ is proved in [13]. The numerator of $g(x)$ is the well-know Fichera number. The concept of Fichera number is very important when we deal with degenerate elliptic problems with boundary characteristic degenerate. It indicates whether we should impose boundary condition in such case. This fact was first observed by M.V.Keldyš in [15] and developed by Fichera in [7, 8]. The Fichera number also affects the regularities of the solutions up to the boundary, see [13]. For more details of Fichera number, refer to [20].

Remark 1.2. *It might be hard to understand that the nonlinearity of $f(x, u)$ should be related to $g(x)$. We can take equation (1.1) for instance to explain why this happens. In this situation, $\phi = y, f = u^\alpha$. It is easy to see $g(x, 0) = a - 1$ by a direct computation. Theorem 1.1 tells us that the existence of non-trivial positive solution depends on the nonlinear power α which is involved in a. When we use blow up method to get a priori estimates of (1.7), one of the limit cases is (1.1) as the blow up point approaching the boundary. It is nature that the the nonlinearity of $f(x, u)$ should be related to $g(x)$ if we want to get the a priori bounds.*

The proof of Theorem 1.2 mainly follows the blow up method used in [12]. The mainly difficulty we encounter is the case when the blow-up point approach to the boundary. This case becomes complicated with the degeneracy of the equation on the boundary and without boundary condition. We should take a suitable transformation of coordinates to make the limit equation exists and establish some regularities estimates up to the boundary to guarantee the point-wise convergence.

The present paper is organized as follows. In Section 2, we establish some lemmas which are similar to Lemma 2.3 and Lemma 2.4 in [2], and necessary for utility of the moving plane method. In Section 3, we shall use the moving plane method to prove Theorem 1.1. In Section 4, as an application of Theorem 1.1, we derive a priori bounds for positive solutions of some semi-linear degenerate elliptic equations.

2 Preliminary Results

In this section we collect some preliminary results which will be needed for our later analysis.

Suppose that u solves (1.1). Set $x_{n+1} = 2\sqrt{y}$ and $\bar{u}(x_1, ..., x_{n+1}) = u(x_1, ..., x_n, \frac{x_{n+1}^2}{4})$. Then (1.1) is reduced to

$$\Delta_{n+1} \bar{u} + \frac{2a - 1}{x_{n+1}} \partial_{n+1} \bar{u} + \bar{u}^\alpha = 0 \text{ in } R_+^{n+1}. \tag{2.1}$$

Noting that $u \in C^2(\overline{R_+^{n+1}})$, we must have

$$\partial_{n+1} \bar{u} = \frac{x_{n+1}}{2} u_y \Rightarrow \partial_{n+1} \bar{u}(x', 0) = 0 \text{ where } x' = (x_1, ..., x_n).$$

This allows us to extend $\bar{u}$ to the lower half-space by

$$\bar{u}(x', x_{n+1}) = \bar{u}(x', -x_{n+1}) \text{ for } x_{n+1} < 0,$$

such that $\bar{u}(x) \in C^2(R^{n+1})$ with $\partial_{n+1}\bar{u}(x',0)=0$.

Consider the following elliptic operator

$$L(u)=\sum_{i=1}^{n+1} a_{ij}(x)\partial_{ij}u+\sum_{i=1}^{n} b_i(x)\partial_i u+\frac{a(x)}{x_{n+1}}\partial_{n+1}u.$$

All the coefficients $a_{ij}(x), b_i(x), a(x) \in C(R^{n+1})$, $a(x) \geq 0$ and (a_{ij}) is a positive definite matrix. Then we shall have the following two lemmas.

Lemma 2.1. *Suppose that $u \in C^2(B_1)\cap C(\bar{B}_1)$ with $\partial_{n+1}u(x',0)=0$ satisfies that*

$$-L(u) \geq 0 \ \text{in } B_1.$$

Then either u is a constant or u can not attain its minimum in B_1.

Proof. It suffices to prove the second case. Without loss of generality, we may assume that u attains its minimum at the origin.

Denote

$$B_r(P)=\{x\in B_1 \big| x'^2+(x_{n+1}-r)^2 \leq r^2\}, r<\frac{1}{2}, \text{ where } P=(0',r)\in R^{n+1}_+.$$

$B_{\frac{r}{2}}(P)$ has the same center as $B_r(P)$ but half radius. Set $\Sigma = B_r(P)\backslash B_{\frac{r}{2}}(P)$. We consider

$$h(x)=1-e^{-\beta[x'^2+(x_{n+1}-r)^2-r^2]} \text{ in } \Sigma.$$

Then we have

$$\begin{aligned} L(h) &= \sum_{i=1}^{n+1} a_{ij}(x)\partial_{ij}h+\sum_{i=1}^{n} b_i(x)\partial_i h+\frac{a(x)}{x_{n+1}}\partial_{n+1}h \\ &\leq e^{-\beta[x'^2+(x_{n+1}-r)^2-r^2]}\left\{-c_1\beta^2(x'^2+(x_{n+1}-r)^2)+c_2\beta(1+r)-\frac{2a(x)\beta r}{x_{n+1}}\right\} \\ &\leq -c_0 \text{ in } \Sigma, \quad \text{since } a(x)\geq 0. \end{aligned} \tag{2.2}$$

for some positive constant $c_0>0$ if we take β large enough. Now let $v=u+\epsilon h$, then $Lv<0$. This implies that v must attain its minimum on the boundary of Σ. Consider v on the boundary of Σ,

(i) on $\partial B_r(P)$, noting that $h|_{\partial B_r(P)}=0$, $u|_{\partial B_r(P)}\geq u(0)$, then we have

$$u+\epsilon h|_{\partial B_r(P)} \geq u(0).$$

(ii) on $\partial B_{\frac{r}{2}}(P)$, there exists $\delta>0$ such that $u|_{\partial B_{\frac{r}{2}}(P)}\geq u(0)+\delta$. Thus we can choose ϵ small enough such that

$$u+\epsilon h|_{\partial B_{\frac{r}{2}}(P)} \geq u(0)+\frac{\delta}{2}.$$

This means $u+\epsilon h\geq u(0)$ in Σ, which implies

$$\partial_{n+1}u(0)\geq -\epsilon\partial_{n+1}h(0)>0.$$

This contradicts to $\partial_{n+1}u(x',0)=0$. This ends the proof of the present lemma. $\square$

Lemma 2.2. *If $u(x)\in C^2(B_1)\cap C^1(\bar{B}_1)$ with $\partial_{n+1}u(x',0)=0$ satisfies that*

$$-L(u)\geq 0 \ \text{in } B_1. \tag{2.3}$$

If u attains its minimum at $x^0\in\partial B_1$, then either $u\equiv const$ or

$$-\frac{\partial u}{\partial n}|_{x=x^0}>0, \ n \text{ is the outward normal to } \partial B_1 \text{ at } x^0.$$

Proof. Assume that u is not a constant. By Lemma 2.1, $u(x)$ can not attain its minimum in B_1. If u attains the minimum at $x^0 \in \partial B_1 \backslash \{x_{n+1} = 0\}$, it is the immediate consequence of the standard Hopf' lemma. Without loss of any generality, we assume that v attains its minimum at $x^0 = (1, 0, ..., 0)$. Set

$$h(x) = e^{-\beta x^2} - e^{-\beta} \qquad \text{in } B_1 \backslash B_{\frac{1}{2}}.$$

Then $h(x) \geq 0$, $h|_{\partial B_1} = 0$ and

$$\begin{aligned} L(h) &= \exp\{-\beta x^2\}\{4\beta^2 a_{ij} x_i x_j - 2\beta a_{ii} - 2b_i \beta x_i - 2a(x)\beta\} \\ &\geq \exp\{-\beta x^2\}\{c_0 \beta^2 x^2 - c_1 \beta |x| - c_2\} > 0, \end{aligned}$$

if we choose β large enough. Choosing $\epsilon > 0$ small enough, one can get

$$L(v - \epsilon h) < 0 \text{ in } B_1 \backslash B_{\frac{1}{2}}, \ u - \epsilon h \geq u(x^0) \text{ on } \partial B_1 \cup \partial B_{\frac{1}{2}}.$$

An application of the maximum principle to $u - \epsilon h$ yields

$$\frac{\partial u}{\partial x_1}|_{x=x^0} \leq \epsilon \frac{\partial h}{\partial x_1}|_{x=x^0} < 0.$$

This completes the proof of the present lemma. $\square$

Turn back to (2.1) and consider

$$v(x) = \frac{1}{|x|^{n+2a-2}} \bar{u}(\frac{x}{|x|^2}).$$

By a direct computation with $\tau = n + 2a + 2 - \alpha(n + 2a - 2)$, v solves,

$$\Delta_{n+1} v + \frac{2a-1}{x_{n+1}} \partial_{n+1} v + |x|^{-\tau} v^{\alpha} = 0 \text{ in } R^{n+1} \backslash \{0\}, \ \partial_{n+1} v(x', 0) = 0 \text{ for } x' \neq 0. \tag{2.4}$$

From the definition of v, we will have the following asymptotic behavior at ∞

$$\begin{cases} v(x) = \dfrac{a_0}{|x|^{n+2a-2}} + \displaystyle\sum_{i=1}^{n+1} \dfrac{a_i x_i}{|x|^{n+2a}} + O(\dfrac{1}{|x|^{n+2a}}), \\ \partial_i v(x) = -\dfrac{(n+2a-2)a_0 x_i}{|x|^{n+2a}} + O(\dfrac{1}{|x|^{n+2a}}), \\ \partial_{ij} v(x) = O(\dfrac{1}{|x|^{n+2a}}), \text{ with } a_0 > 0. \end{cases} \tag{2.5}$$

Next we generalize the important lemmas which are essential for the application of moving plane method to (1.4) in [2] to the equation (1.1) studied in the present paper. Denote

$$\Sigma_\lambda = \{x \in R^{n+1} | x_1 < \lambda\}, x^\lambda = (2\lambda - x_1, x_2, ..., x_{n+1}).$$

Then there hold the following lemmas

Lemma 2.3. *Let v be a function in a neighborhood of infinity satisfying the asymptotic expansion (2.5). Then there exist two positive constants R, λ_1 such that, if $\lambda \geq \lambda_1$,*

$$v(x) > v(x^\lambda), \quad \text{for } x \in \Sigma_\lambda, |x| > R, \lambda \geq \lambda_1.$$

Proof. In view of (2.5),

$$v(x) - v(x^\lambda) = a_0(\frac{1}{|x|^{n+2a-2}} - \frac{1}{|x^\lambda|^{n+2a-2}}) + \sum_{j=1}^{n+1} a_j x_j (\frac{1}{|x|^{n+2a}} - \frac{1}{|x^\lambda|^{n+2a}})$$

$$+ \quad \frac{2a_1(x_1-\lambda)}{|x^\lambda|^{n+2a}} + O(\frac{1}{|x|^{n+2a}}), \tag{2.6}$$

(1) if $|x^\lambda| \geq 2|x|$, there holds

$$v(x) - v(x^\lambda) \geq \frac{c_0}{|x|^{n+2a-2}} - \frac{c_1}{|x|^{n+2a-1}} > 0 \text{ if } |x| > R, \text{ for sufficiently large } R,$$

(2) if $|x^\lambda| < 2|x|$, there holds

$$a_0(\frac{1}{|x|^{n+2a-2}} - \frac{1}{|x^\lambda|^{n+2a-2}}) \geq \frac{a_0}{|x|^{n+2a-3}}(\frac{1}{|x|} - \frac{1}{|x^\lambda|}) \geq \frac{c_0(|x^\lambda| - |x|)}{|x|^{n+2a-1}}.$$

Hence

$$v(x) - v(x^\lambda) \geq \frac{c_1(|x^\lambda| - |x|)}{|x|^{n+2a-1}} - \frac{c_2}{|x|^{n+2a}}.$$

If $|x^\lambda| - |x| > \frac{c_2}{c_1}\frac{1}{|x|}$, it follows that $v(x) > v(x^\lambda)$.

If $|x^\lambda| - |x| \leq \frac{c_2}{c_1}\frac{1}{|x|}$, this implies $x_1 \geq \frac{\lambda}{2}$. By asymptotic expansion of v at ∞, $v_{x_1} < 0$ if we choose λ large enough, thus $v(x) - v(x^\lambda) > 0$. □

Lemma 2.4. *Suppose that v satisfies the assumption of Lemma 2.3 and*

$$\begin{cases} \Delta_{n+1}(v(x) - v(x^{\lambda_0})) + \frac{2a-1}{x_{n+1}}\partial_{n+1}(v(x) - v(x^{\lambda_0})) \leq 0 \text{ in } \Sigma_{\lambda_0} \cap \{|x| > R\}, \\ v(x) > v(x^{\lambda_0}), \ \forall x \in \Sigma_{\lambda_0} \cap \{|x| > R\}, \end{cases} \tag{2.7}$$

with $\partial_{n+1}v(x', 0) = 0$ and $a \geq \frac{1}{2}$. Then there exist two constants ϵ and $S > 0$ such that

(i) $v_{x_1} < 0$ in $|x_1 - \lambda_0| < \epsilon$ and $|x| > S$,

(ii) $v(x) > v(x^\lambda)$ in $x_1 \leq \lambda_0 - \frac{\epsilon}{2} \leq \lambda$ and $|x| > S$ for all $x \in \Sigma_\lambda$ with $|\lambda - \lambda_0| \leq o(\epsilon)$.

Proof. With $w(x) = v(x) - v(x^{\lambda_0})$, it is easy to see $w(x)|_{x_1=\lambda_0} = 0$ and that

$$\begin{cases} \Delta w + \frac{2a-1}{x_{n+1}}\partial_{n+1}w \leq 0, \quad \text{in } \Sigma_{\lambda_0} \cap \{|x| > R\}, \\ w(x) > 0, \partial_{n+1}w(x', 0) = 0 \text{ in } \Sigma_{\lambda_0} \cap \{|x| > R\}. \end{cases} \tag{2.8}$$

We claim: there exists $\delta > 0$ so small that

$$w(x) > \frac{\delta(\lambda_0 - x_1)}{|x - \lambda_0\mathbf{e}_1|^{n+2a}} \qquad \text{in } \Sigma_{\lambda_0} \cap \{|x| > R + 1\}. \tag{2.9}$$

Here $\mathbf{e}_i$ means the vector whose i-th coordinate is 1 and others are 0. By Lemma 2.2, we see that $w_{x_1} < 0$ on $\{|x| = R + 1\} \cap \{x_1 = \lambda_0\}$ which implies for some small $\bar{\delta}, k_0 > 0$,

$$w(x) > \frac{k_0(\lambda_0 - x_1)}{|x - \lambda_0\mathbf{e}_1|^{n+2a}} \text{ on } \{|x| = R + 1\} \cap \{\lambda_0 - \bar{\delta} \leq x_1 < \lambda_0\}.$$

We shall notice that $w(x) \geq c_0 > 0$ on $\{|x| = R + 1\} \cap \{x_1 \leq \lambda_0 - \bar{\delta}\}$, thus

$$w(x) > \frac{\delta(\lambda_0 - x_1)}{|x - \lambda_0\mathbf{e}_1|^{n+2a}} \text{ on } \{|x| = R + 1\} \cap \{x_1 < \lambda_0\}.$$

Denote

$$h(x) = \frac{\lambda_0 - x_1}{|x - \lambda_0\mathbf{e}_1|^{n+2a}}.$$

It is easy to see that

$$\Delta_{n+1}h + \frac{2a-1}{x_{n+1}}\partial_{n+1}h = 0 \text{ in } R^{n+1}\backslash\{\lambda_0\mathbf{e}_1\}.$$

Therefore (2.9) is proved by the maximum principle. In particular,

$$w_{x_1}(\lambda_0, x'') = 2v_{x_1}(\lambda_0, x'') < -\delta/|x''|^{n+2a}, \text{ where } x'' = (x_2, x_3, ..., x_{n+1}).$$

Combining this with the asymptotic expansion yields

$$\begin{aligned} v_{x_1}(\lambda_0 + h, x'') &\leq v_{x_1}(\lambda_0, x'') + \frac{C|h|}{|x|^{n+2a}} \\ &\leq -\frac{1}{2}\delta\frac{1}{|x''|^{n+2a}} + \frac{C|h|}{|x|^{n+2a}} \\ &\leq -\frac{1}{4}\delta\frac{1}{|x|^{n+2a}}, \end{aligned}$$

as $|h| < \frac{\delta}{4C}$ and $|x|$ is large. This proves the first part of the present lemma.

As for the second part, from the asymptotic expansion and the result of the first part, it follows that

$$v(2\lambda_0 - x_1, x'') - v(2\lambda - x_1, x'') \geq \frac{-c(\lambda_0 - \lambda)}{|x - \lambda_0 e_1|^{n+2a}}(\lambda_0 - x_1 + c),$$

as $x_1 < \lambda_0$ and $|x|$ is large. Hence,

$$\begin{aligned} & v(x_1, x'') - v(2\lambda - x_1, x'') \\ = \; & (v(x_1, x'') - v(2\lambda_0 - x_1, x'')) + (v(2\lambda_0 - x_1, x'') - v(2\lambda - x_1, x'')) \\ \geq \; & \frac{\delta(\lambda_0 - x_1)}{|x - \lambda_0 e_1|^{n+2a}} - \frac{c(\lambda_0 - \lambda)}{|x - \lambda_0 e_1|^{n+2a}}(\lambda_0 - x_1 + c) \\ = \; & \frac{[\delta - c(\lambda_0 - \lambda)](\lambda_0 - x_1) - c^2(\lambda_0 - \lambda)}{|x - \lambda_0 e_1|^{n+2a}} > 0, \end{aligned}$$

if $x_1 \leq \lambda_0 - \frac{\epsilon}{2}$ and $|\lambda - \lambda_0|$ is sufficiently small compared to ϵ. This completes the proof of the present lemma. $\square$

In all the above arguments, the discussion is always carried out outside a neighborhood of the origin. Now let us investigate the behavior of v in this neighborhood. The following idea mainly comes from [17] and [5].

Lemma 2.5. *Suppose that $v \in C^2(B_1\backslash\{0\}) \cap C(\bar{B}_1\backslash\{0\})$ is a positive solution to the following problem with $n + 2a > 2$,*

$$L(v) = \Delta_{n+1}v + \frac{2a-1}{x_{n+1}}\partial_{n+1}v \leq 0 \text{ in } B_1\backslash\{0\} \text{ with } \partial_{n+1}v(x', 0) = 0. \tag{2.10}$$

Then there holds

$$v(x) \geq \inf_{\partial B_1} v, \qquad \forall x \in B_1\backslash\{0\}.$$

Proof. Set $\inf_{\partial B_1} v = m_1$. Consider $h_s(x)$, with $0 < s < 1$ and a suitable constant $l(s)$,

$$h_s(x) = m_1 + l(s)(\frac{1}{|x|^{n+2a-2}} - 1), h_s|_{\partial B_1} = m_1, h_s|_{\partial B_s} = -1.$$

From the definition of $h_s(x)$, there holds

$$L(v - h_s) \leq 0 \text{ in } B_1\backslash B_s, \quad v - h_s|_{\partial B_1} \geq 0, \quad v - h_s|_{\partial B_s} \geq 1.$$

By the maximum principle we have for any fixed x

$$v(x) \geq h_s(x) = m_1 + l(s)(\frac{1}{|x|^{n+2a-2}} - 1), \quad \text{if } s \text{ is small} .$$

It is easy to see

$$l(s) = -\frac{(m_1+1)s^{n+2a-2}}{1 - s^{n+2a-2}} \to 0 \text{ as } s \to 0.$$

Thus passing to the limit $s \to 0$, we have proved $v(x) \geq m_1$. This finishes the proof of the present lemma. □

3 The proof for Theorem 1.1

Now we can prove Theorem 1.1.

Proof of Theorem 1.1: Set

$$\lambda_0 = \inf_{\lambda}\{\lambda \big| v(x) > v(x^{\lambda'}) \ \ \forall x \in \Sigma_{\lambda'}, \lambda' \geq \lambda\}.$$

By Lemma 2.3 and Lemma 2.5, we have $|\lambda_0| < \infty$. Also we can claim that

$$v(x) = v(x^{\lambda_0}), \quad \forall x \in \Sigma_{\lambda_0} \backslash \{0\}. \tag{3.1}$$

Without loss of generality, we may assume $\lambda_0 > 0$. Since for the case $\lambda_0 = 0$, we can start the moving plane from $-\infty$ and stop at $x_1 = \lambda_1$. If $\lambda_1 < 0$, we can prove $v(x) = v(x^{\lambda_1})$ by the same arguments as we do in the case $\lambda_0 > 0$. Otherwise $\lambda_1 = 0$, the claim (3.1) holds immediately. Now we turn to prove the claim (3.1) for $\lambda_0 > 0$. If (3.1) is false, by Lemma 2.1, one gets

$$v(x) > v(x^{\lambda_0}), \quad \forall x \in \Sigma_{\lambda_0} \backslash \{0\}.$$

Also from the definition of λ_0 and Lemma 2.4, one can choose $\lambda_k \uparrow \lambda_0$ as $k \to \infty$ such that

$$\emptyset \neq \sigma_k = \{x \in \Sigma_{\lambda_k} \backslash \{0\} \big| v(x) \leq v(x^{\lambda_k})\} \subset B_R.$$

Noting Lemma 2.5 and the continuity of v, we can see that

$$v(x) > v(x^{\lambda_k}) \text{ in } B_r \backslash \{0\}, \ B_r \subset \Sigma_{\lambda_k},$$

if we choose r small enough and k large enough. This implies that $\sigma_k \subset B_R \backslash B_r$. Set $w_k(x) = v(x) - v(x^{\lambda_k})$ and $w(x) = v(x) - v(x^{\lambda_0})$. It is easy to see that $\exists x^k \in \sigma_k$ such that $w_k(x^k) = \inf w_k(x)$. Next we consider $x^\infty = \lim\limits_{k \to \infty} x^k$ in two cases

(1) $x^\infty \in \Sigma_{\lambda_k}$: we have $w(x^\infty) = \lim\limits_{k\to\infty} w_k(x^k) \leq 0$ which is a contradiction.

(2) $x^\infty \in \partial\Sigma_{\lambda_k}$: we have $\partial_{x_1} w(x^\infty) = \lim\limits_{k\to\infty} \partial_{x_1} w_k(x^k) = 0$ which is a contradiction too.

This proves the assertion (3.1).

If $\alpha < \frac{n+2a+2}{n+2a-2}$, we have $\tau > 0$. To prove the radial symmetry of v, one should take a transformation. Set

$$\bar{v}(x', x_{n+1}, x_{n+2}) = v(x', \sqrt{x_{n+1}^2 + x_{n+2}^2}).$$

It follows that,

$$\Delta_{n+2}\bar{v} + \frac{2a-2}{x_{n+2}}\partial_{n+2}\bar{v} + |x|^{-\tau}\bar{v}^\alpha = 0, \text{ in } R^{n+2}, \partial_{n+2}\bar{v}(x', x_{n+1}, 0) = 0. \tag{3.2}$$

There is a singularity at 0, and hence λ_0 must be 0. Notice that (3.2) is rotation invariant about x', x_{n+1}. We have

$$v(x', x_{n+1}) = \bar{v}(x', x_{n+1}, 0) = \bar{v}(\bar{x}', \bar{x}_{n+1}, 0) = v(\bar{x}', \bar{x}_{n+1}), \text{ if } |x'|^2 + x_{n+1}^2 = |\bar{x}'|^2 + \bar{x}_{n+1}^2.$$

This implies that

$$\bar{u}(x', x_{n+1}) = \bar{u}(\bar{x}', \bar{x}_{n+1}), \text{ if } |x'|^2 + x_{n+1}^2 = |\bar{x}'|^2 + \bar{x}_{n+1}^2.$$

If we take another transformation such as

$$v_b(x) = \frac{1}{|x|^{n+2a-2}} \bar{u}_b\left(\frac{x}{|x|^2}\right), \text{ here } b_{n+1} = 0,$$

where $\bar{u}_b(x) = \bar{u}(x - b)$. Repeating the above arguments, similarly we have

$$\bar{u}(x', x_{n+1}) = \bar{u}(\bar{x}', \bar{x}_{n+1}), \text{ if } |x' + b'|^2 + x_{n+1}^2 = |\bar{x}' + b'|^2 + \bar{x}_{n+1}^2.$$

In fact, b' can be chosen arbitrarily, thus $\bar{u}$ must be a constant. This means that $\bar{u} \equiv 0$.

Now we consider the case $\alpha = \frac{n+2a+2}{n+2a-2}$ or $\tau = 0$. By the same arguments as we did in the case $\tau > 0$, there exists $\lambda = (\lambda_1, ..., \lambda_{n+1})$ such that

$$\bar{v}(x', x_{n+1}, 0) = v(x', x_{n+1}) = v(\bar{x}', \bar{x}_{n+1}) = \bar{v}(\bar{x}', \bar{x}_{n+1}, 0), \text{ if } \sum_{i=1}^{n+1} |x_i - \lambda_i|^2 = \sum_{i=1}^{n+1} |\bar{x}_i' - \lambda_i|^2. \quad (3.3)$$

In fact, λ_{n+1} must be 0. Otherwise, it follows that

$$v(x', 2\lambda_{n+1} - x_{n+1}) = v(x', x_{n+1}) = v(x', -x_{n+1}).$$

It shows that for the fixed x', v is periodic with respect to x_{n+1} with period $2\lambda_{n+1}$. This means that v must vanish which is impossible. For $\lambda' = (\lambda_1, ..., \lambda_n)$, we have two cases.

(1) $\lambda' = 0$: noting $\bar{u}(x) = \frac{1}{|x|^{n+2a-2}} v(\frac{x}{|x|^2})$, $\bar{u}(x)$ is radially symmetric with respect to the origin.

(2) $\lambda' \neq 0$: This means that 0 is not the symmetric center of v, v must be C^2 at 0. In other words, $\bar{u}(x)$ has the similar asymptotic behavior at ∞ as $v(x)$. This allows us to apply the moving plane method to $\bar{u}(x)$ directly to obtain that $\bar{u}(x)$ is radially symmetric with respect to some point $b \in R^{n+1}, b_{n+1} = 0$.

The above arguments show that $\bar{u}(x)$ is radially symmetric with respect to a point $b \in \{b_{n+1} = 0\}$. Now we can follow the arguments of Section 3 in [3], then we can complete the proof of Theorem 1.1.

Comparing Theorem 1.1 with (1.5), we can regard (1.1) as an equation defined in dimension $n + 2a$. Therefore, we can consider the following more general equation

$$\begin{cases} \sum_{i=1}^{m} y_i u_{y_i y_i} + \sum_{i=1}^{m} a_i u_{y_i} + \Delta_x u + u^\alpha = 0 \text{ in } R_+^{m,n} = \{(x, y) \big| x \in R^n, y_i \in R_+^1, i = 1, ..., m\}, \\ u \geq 0 \text{ in } R_+^{m,n} \text{ and } u \in C^2(\overline{R_+^{m,n}}). \end{cases} \quad (3.4)$$

Theorem 3.1. *Let $u(x, y)$ be a nonnegative solution of (3.4) with constants $a_i > 1, i = 1, ..., m$. Then with $a = \sum_{i=1}^{m} a_i$*

(1) for $1 < \alpha < \frac{n+2a+2}{n+2a-2}$, $u \equiv 0$.

(2) for $\alpha = \frac{n+2a+2}{n+2a-2}$, $$u_{t,x_0}(x, y) = \left(\frac{t\sqrt{(n + 2a)(n + 2a - 2)}}{t^2 + 4\sum_{i=1}^{m} y_i + |x - x_0|^2} \right)^{\frac{n+2a-2}{2}},$$

for some $x_0 \in R^n$ and $t \geq 0$.

The proof of Theorem 3.1 is just the same as the proof of Theorem 1.1, as we can easily establish the similar lemmas as in Section 2 for (3.4). Thus we omit the details here.

4 An application to a priori estimates of semi-linear degenerate elliptic equations

The proof for Theorem 1.2: our proof is by contradiction and uses a scaling argument reminiscent to that used in the theory of Minimal Surfaces, also refer to [12]. If (1.13) is false, we can get a sequence $u^k \in C^2(\Omega) \cap L^\infty(\Omega)$ such that

$$|u^k|_{L^\infty} = M_k \to \infty \text{ as } k \to \infty. \tag{4.1}$$

Hence, we can find $x^k \in \Omega \to \bar{x} \in \bar{\Omega}$ as $k \to \infty$ such that $u^k(x^k) \geq \frac{M_k}{2}$. Next we shall distinguish two cases to investigate.

Case 1: $\bar{x} \in \Omega$. With $y = \frac{x-x^k}{\mu_k}$ define the scaled function

$$v^k(y) = \mu_k^{\frac{2}{\alpha-1}} u^k(x) \text{ where } \mu_k^{\frac{2}{\alpha-1}} M_k = 1. \tag{4.2}$$

For large k, $v^k(y)$ is well defined in $B_{\frac{d}{\mu_k}}(0)$ where $2d = \mathrm{dist}(\bar{x}, \partial\Omega)$,

$$\sup_{y \in B_{\frac{d}{\mu_k}}(0)} v^k(y) = 1, v^k(0) \geq \frac{1}{2}, \tag{4.3}$$

Moreover, $v^k(y)$ satisfies

$$a_k^{ij} \frac{\partial^2 v^k}{\partial y_i \partial y_j} + \mu_k b_k^i \frac{\partial v^k}{\partial y_i} + \mu_k^{\frac{2\alpha}{\alpha-1}} f(\mu_k y + x^k, \mu_k^{-\frac{2}{\alpha-1}} v^k) = 0, \text{ in } B_{\frac{d}{\mu_k}}(0) \tag{4.4}$$

where $a_k^{ij}(y) = a^{ij}(\mu_k y + x^k), b_k^i(y) = b^i(\mu_k y + x^k)$. Noting that $y \in B_{\frac{d}{\mu_k}}(0)$ which implies $\mu_k y + x^k \in B_d(x^k)$ and $\mathrm{dist}(B_d(x^k), \partial\Omega) > \frac{d}{2}$ for k large enough, one has (4.4) is uniformly elliptic in $B_{\frac{d}{\mu_k}}(0)$. From (1.11), we see that

$$\lim_{k\to\infty} |\mu_k^{\frac{2\alpha}{\alpha-1}} f(\mu_k y + x^k, \mu_k^{-\frac{2}{\alpha-1}} v^k) - h(\mu_k y + x^k)(v^k(y))^\alpha| = 0.$$

Therefore, given any R such that $B_R(0) \subset B_{\frac{d}{\mu_k}}(0)$, we can, by elliptic L^p estimates, find uniform bounds for $\|v^k\|_{W^{2,p}(B_R(0))}$. Choosing p large, we obtain by Sobolev embedding theorem that $\|v^k\|_{C^{1,\beta}(B_R(0))}$, $0 < \beta < 1$, is also uniformly bounded. Passing to the limit $k \to \infty$ gives $v^k \to v$ and v solves

$$a^{ij}(\bar{x}) \frac{\partial^2 v}{\partial y_i \partial y_j} + h(\bar{x}) v^\alpha = 0, \text{ in } R^2, v(0) \geq \frac{1}{2}. \tag{4.5}$$

By performing a rotation and stretching of coordinates, (4.5) is reduced to

$$\Delta v + v^\alpha = 0 \text{ in } R^2. \tag{4.6}$$

Suppose v is a non-trivial non-negative solution of (4.6). Let $\tilde{v}(y_1, y_2, y_3) = v(y_1, y_2)$. Then

$$\Delta \tilde{v} + \tilde{v}^\alpha = 0 \text{ in } R^3 \tag{4.7}$$

Noting $\alpha < \frac{3+2a}{2a-1} < \frac{3+2}{3-2} = 5$ and the results of [11], we must have $\tilde{v} \equiv 0$ which contradicts to $\tilde{v}(0, y_3) \geq \frac{1}{2}$.

Case 2: $\bar{x} \in \partial\Omega$. This is quite different from Case 1. Without loss of generality, we may assume that

$$\partial_1 \phi(\bar{x}) = 0, \partial_2 \phi(\bar{x}) \neq 0.$$

From (1.9) it follows

$$0 = a^{ij}\partial_i\phi\partial_j\phi(\bar{x}) = a^{22}(\partial_2\phi)^2 \Rightarrow a^{22}(\bar{x}) = 0.$$

Hence $a^{11}(\bar{x}) > 0$ follows immediately from (1.10). Denote

$$y_1 = x_1, y_2 = \phi(x), \quad \forall x \in B_d(\bar{x}) \cap \bar{\Omega}, \ d \text{ small enough.}$$

Therefore, in the new coordinates (y_1, y_2), (1.7) can be written as for some small δ

$$\tilde{a}^{22}\frac{\partial^2 u^k}{\partial y_2^2} + \tilde{a}^{11}\frac{\partial^2 u^k}{\partial y_1^2} + 2\tilde{a}^{12}\frac{\partial^2 u^k}{\partial y_1 \partial y_2} + \tilde{b}^1\frac{\partial u^k}{\partial y_1} + \tilde{b}^2\frac{\partial u^k}{\partial y_2} + f(y, u^k) = 0, \text{ in } B_\delta(y^k) \cap \{y_2 > 0\} \quad (4.8)$$

where

$$\tilde{a}^{22} = a^{ij}\partial_i\phi\partial_j\phi, \tilde{a}^{11} = a^{11}, \tilde{a}^{12} = a^{1j}\partial_j\phi, \tilde{b}^1 = b^1, \tilde{b}^2 = b^j\partial_j\phi + a^{ij}\partial_{ij}\phi. \quad (4.9)$$

Set

$$p_1 = \frac{y_1 - y_1^k}{\mu_k}, p_2 = \frac{y_2 - y_2^k}{\mu_k^2}, v^k(p) = \mu_k^{\frac{2}{\alpha-1}} u^k(y) \text{ with } \mu_k^{\frac{2}{\alpha-1}} M_k = 1.$$

Then

$$\begin{aligned} \mu_k^{-2}\tilde{a}^{22}\frac{\partial^2 v^k}{\partial p_2^2} + \tilde{a}^{11}\frac{\partial^2 v^k}{\partial p_1^2} \quad &+ \quad 2\mu_k^{-1}\tilde{a}^{12}\frac{\partial^2 v^k}{\partial p_1 \partial p_2} + \mu_k\tilde{b}^1\frac{\partial v^k}{\partial p_1} \quad (4.10) \\ &+ \quad \tilde{b}^2\frac{\partial v^k}{\partial p_2} + \mu_k^{\frac{2\alpha}{\alpha-1}} f(p, \mu_k^{-\frac{2}{\alpha-1}} v^k) = 0 \text{ in } B_{\frac{\delta}{\mu_k}}(0) \cap \{p_2 > -\frac{y_2^k}{\mu_k^2}\}. \end{aligned}$$

Set $H_k = B_{\frac{\delta}{\mu_k}}(0) \cap \{p_2 > -\frac{y_2^k}{\mu_k^2}\}$. Then we will have the following lemma

Lemma 4.1. *In the region considered, one has*

$$\tilde{a}^{11} \geq c_0 > 0, \tilde{a}^{12} = A^{12}(p)\mu_k^2(p_2 + \frac{y_2^k}{\mu_k^2}), \tilde{a}^{22} = A^{22}(p)\mu_k^2(p_2 + \frac{y_2^k}{\mu_k^2}), \quad (4.11)$$

where

$$A^{12}, A^{22} \in C^1(\bar{H}_k), A^{22}(p_1, -\frac{y_2^k}{\mu_k^2}) > 0, \frac{\tilde{b}^2(p_1, -\frac{y_2^k}{\mu_k^2})}{A^{22}(p_1, -\frac{y_2^k}{\mu_k^2})} > 2. \quad (4.12)$$

Proof. Noting $\tilde{a}^{22} = a^{ij}\partial_i\phi\partial_j\phi = 0$ on $\{y_2 = 0\}$, we get

$$\begin{aligned} \tilde{a}^{22}(\bar{y}) &= \int_0^1 \frac{d(a^{ij}\partial_i\phi\partial_j\phi)(\bar{y}_1, t\bar{y}_2)}{dt} dt = \bar{y}_2 \int_0^1 \partial_{y_2}(a^{ij}\partial_i\phi\partial_j\phi)(\bar{y}_1, t\bar{y}_2) dt \\ &= A^{22}\mu_k^2(\bar{p}_2 + \frac{y_2^k}{\mu_k^2}), \text{ where } A^{22} = \int_0^1 \partial_{y_2}(a^{ij}\partial_i\phi\partial_j\phi)(\bar{y}_1, t\bar{y}_2) dt. \quad (4.13) \end{aligned}$$

From (1.9), we see that $\nabla(\tilde{a}^{22}) \neq 0$ on $\{y_2 = 0\}$ which implies that $\partial_{y_2}\tilde{a}^{22}(y_1, 0) > 0$ or $A^{22}(p_1, -\frac{y_2^k}{\mu_k^2}) > 0$ in the region considered. The C^1 property of A^{22} follows from the C^2 property of a^{ij}, ϕ immediately. The last term in (4.12) follows from (1.12).

□

Dividing both sides of (4.10) by A^{22}, one can get in H_k

$$(p_2 + \frac{y_2^k}{\mu_k^2})\frac{\partial^2 v^k}{\partial p_2^2} + \bar{a}^{11}\frac{\partial^2 v^k}{\partial p_1^2} \quad + \quad 2\mu_k(p_2 + \frac{y_2^k}{\mu_k^2})\bar{a}^{12}\frac{\partial^2 v^k}{\partial p_1 \partial p_2}$$

$$+\quad \mu_k \bar{b}^1 \frac{\partial v^k}{\partial p_1} + \bar{b}^2 \frac{\partial v^k}{\partial p_2} + \mu_k^{\frac{2\alpha}{\alpha-1}} g(p, \mu_k^{-\frac{2}{\alpha-1}} v^k) = 0. \qquad (4.14)$$

where

$$\bar{a}^{11} = \frac{\tilde{a}^{11}}{A^{22}}, \bar{a}^{12} = \frac{A^{12}}{A^{22}}, \bar{b}^i = \frac{\tilde{b}^i}{A^{22}}, \bar{f} = \frac{f}{A^{22}}.$$

We must take care of the limit of $\frac{y_2^k}{\mu_k^2}$.

Case 2.1: $\lim_{k\to} \frac{y_2^k}{\mu_k^2} = \infty$. We take $q_1 = p_1, q_2 = 2\sqrt{p_2 + \frac{y_2^k}{\mu_k^2}} - 2\sqrt{\frac{y_2^k}{\mu_k^2}}$, then (4.14) changes to

$$\begin{aligned}\frac{\partial^2 v^k}{\partial q_2^2} \quad &+ \quad \bar{a}^{11} \frac{\partial^2 v^k}{\partial q_1^2} + \mu_k \left(q_2 + 2\sqrt{\frac{y_2^k}{\mu_k^2}} \right) \bar{a}^{12} \frac{\partial^2 v^k}{\partial q_1 \partial q_2} \\ &+ \quad \mu_k \bar{b}^1 \frac{\partial v^k}{\partial q_1} + \frac{2\bar{b}^2 - 1}{q_2 + 2\sqrt{\frac{y_2^k}{\mu_k^2}}} \frac{\partial v^k}{\partial q_2} + \mu_k^{\frac{2\alpha}{\alpha-1}} \bar{f}(q, \mu_k^{-\frac{2}{\alpha-1}} v^k) = 0, \text{ in } J_k. \qquad (4.15)\end{aligned}$$

It is important to show that J_k can be chosen arbitrarily large as $k \to \infty$. Since $p \in H_k$, it follows that

$$\begin{aligned}&q_1^2 + \left[\left(\frac{q_2}{2} + \sqrt{\frac{y_2^k}{\mu_k^2}} \right)^2 - \frac{y_2^k}{\mu_k^2} \right]^2 < \frac{\delta^2}{\mu_k^2} \iff q_1^2 + q_2^2 \left(\frac{q_2}{4} + \sqrt{\frac{y_2^k}{\mu_k^2}} \right)^2 < \frac{\delta^2}{\mu_k^2} \\ &\Rightarrow q_2 < \frac{2\delta}{\sqrt{y_2^k}} \text{ noting that } q_2 > -2\sqrt{\frac{y_2^k}{\mu_k^2}}. \qquad (4.16)\end{aligned}$$

From $y_2^k \to 0$ and $\frac{y_2^k}{\mu_k^2} \to \infty$, one can get

$$\frac{\delta^2}{\mu_k^2} = 4 l_k^2 \max{}^2 \left\{ \frac{\delta}{2\sqrt{y_2^k}}, \sqrt{\frac{y_2^k}{\mu_k^2}} \right\}, l_k \to \infty \text{ as } k \to \infty. \qquad (4.17)$$

Since

$$q_1^2 + q_2^2 \left(\frac{q_2}{4} + \sqrt{\frac{y_2^k}{\mu_k^2}} \right)^2 \leq q_1^2 + 4 \max{}^2 \left\{ \frac{\delta}{2\sqrt{y_2^k}}, \sqrt{\frac{y_2^k}{\mu_k^2}} \right\} q_2^2, \qquad (4.18)$$

we can take $J_k = B_{l_k}(0) \cap \{q_2 > -2\sqrt{\frac{y_2^k}{\mu_k^2}}\}$.

Also, we have $v^k(0) \geq \frac{1}{2}$. As for any R, we can choose k large enough such that $B_R(0) \subset J_k$. Thus (4.15) is uniformly elliptic in $B_R(0)$ with uniformly bounded coefficients. This allows us to follow the same steps in Case 1. Namely, passing to limit $k \to \infty$, we have

$$\frac{\partial^2 v}{\partial q_2^2} + \bar{a}^{11}(\bar{x}) \frac{\partial^2 v}{\partial q_1^2} + \frac{h(\bar{x})}{\partial_{y_2}(a^{ij}\phi_i\phi_j)(\bar{x})} v^\alpha = 0 \text{ in } R^2, v(0) \geq \frac{1}{2}, \qquad (4.19)$$

if we notice that for $q \in B_R(0)$,

$$|\mu_k(q_2 + 2\sqrt{\frac{y_2^k}{\mu_k^2}})| + |\frac{2\bar{b}^2 - 1}{q_2 + 2\sqrt{\frac{y_2^k}{\mu_k^2}}}| \leq \mu_k R + 2\sqrt{y_2^k} + \frac{C\mu_k}{\sqrt{y_2^k}} \to 0 \text{ as } k \to \infty. \qquad (4.20)$$

Therefor, (4.19) gives rise a contradiction.

Case 2.2: $\lim\limits_{k\to}\frac{y_2^k}{\mu_k^2}=c<\infty$.

First of all, we establish a lemma of weighted-L^2 estimates. Let $\psi\in C_c^\infty(R^2)$ be a cutoff function with $\psi(p)=1$ as $|p|\le 1/2$ and $\psi=0$ as $|p|\ge 1$. Set $\psi_r(p)=\psi(\frac{p}{r})$.

Lemma 4.2. *Suppose $u\in C^2(R_+^2)\cap L^\infty(R_+^2)$ solves (4.21),*

$$p_2\frac{\partial^2 u}{\partial p_2^2}+B^{11}\frac{\partial^2 u}{\partial p_1^2}+2p_2B^{12}\frac{\partial^2 u}{\partial p_1\partial p_2}+B^1\frac{\partial u}{\partial p_1}+B^2\frac{\partial u}{\partial p_2}+f=0 \text{ in } R_+^2, \tag{4.21}$$

with $B^{ij},B^i\in C^1(\overline{R_+^2}), f\in L^\infty(R_+^2)$ and $B^{11}(p_1,0)\ge c_0>0$. Then for r suitable small, we have

$$\|p_2^{\frac12}\psi_r u_{p_2}\|_{L^2}+\|\psi_r u_{p_1}\|_{L^2}\le C(r,\|\psi_r B^{ij}\|_{C^1},\|\psi_r B^i\|_{C^1},\|\psi_r f\|_{L^\infty},\|\psi_r u\|_{L^\infty}). \tag{4.22}$$

Proof. Set $\eta_\epsilon(p_2)\in C^\infty(R_+^1)$ that

$$\eta_\epsilon(p_2)=\begin{cases}0, & 0<p_2<\epsilon\\ 1, & p_2>2\epsilon,\end{cases} \tag{4.23}$$

with $|D^j\eta_\epsilon|\le C_j\epsilon^{-j}$ for $p_2\in(\epsilon,2\epsilon)$. Denote $\psi_{r,\epsilon}=\psi_r\eta_\epsilon$. Multiplying both sides of (4.21) by $\psi_{r,\epsilon}u$ and integrating by parts, we can get

$$\begin{aligned}
&\int\psi_{r,\epsilon}p_2\left(\frac{\partial u}{\partial p_2}\right)^2+\int B^{11}\psi_{r,\epsilon}\left(\frac{\partial u}{\partial p_1}\right)^2\\
=\;&\int\left(\frac{\partial\psi_{r,\epsilon}}{\partial p_2}+\frac12 p_2\frac{\partial^2\psi_{r,\epsilon}}{\partial p_2^2}-\frac12\frac{\partial(B^2\psi_{r,\epsilon})}{\partial p_2}\right)u^2+\int\left(B^1\psi_{r,\epsilon}-\frac{\partial(B^{11}\psi_{r,\epsilon})}{\partial p_1}\right)u\frac{\partial u}{\partial p_1}\\
-\;&2\int p_2\frac{\partial(B^{12}\psi_{r,\epsilon})}{\partial p_1}u\frac{\partial u}{\partial p_2}-2\int p_2B^{12}\psi_{r,\epsilon}\frac{\partial u}{\partial p_1}\frac{\partial u}{\partial p_2}.
\end{aligned} \tag{4.24}$$

Now we estimate the terms on the right side of (4.24). The first term,

$$\left|\int\frac{\partial\psi_{r,\epsilon}}{\partial p_2}u^2\right|\le\int|\partial_{p_2}\psi_r|\eta_\epsilon u^2+\int|\partial_{p_2}\eta_\epsilon|\psi_r u^2\le C_1+C_2\int_\epsilon^{2\epsilon}\epsilon^{-1}dp_2\le C, \tag{4.25}$$

where C is a constant only depending on the quantities in (4.22). Also

$$\begin{aligned}
\left|\int p_2\frac{\partial\psi_{r,\epsilon}}{\partial p_2^2}u^2\right| &\le \int p_2|\partial_{p_2}^2\psi_r|\eta_\epsilon u^2+2\int p_2|\partial_{p_2}\psi_r\partial_{p_2}\eta_\epsilon|u^2+\int p_2\psi_r|\partial_{p_2}^2\eta_\epsilon|u^2\\
&\le C_1+\epsilon^{-2}\int_\epsilon^{2\epsilon}p_2dp_2\le C
\end{aligned} \tag{4.26}$$

The second term,

$$\left|\int B^1\psi_{r,\epsilon}u\frac{\partial u}{\partial p_1}\right|\le\delta\int\psi_{r,\epsilon}\left(\frac{\partial u}{\partial p_1}\right)^2+\frac{1}{4\delta}\int(B^1)^2\psi_{r,\epsilon}u^2. \tag{4.27}$$

The last term,

$$\begin{aligned}
\left|\int p_2B^{12}\psi_{r,\epsilon}\frac{\partial u}{\partial p_1}\frac{\partial u}{\partial p_2}\right| &\le \int p_2^{\frac32}\psi_{r,\epsilon}|B^{12}|\left(\frac{\partial u}{\partial p_2}\right)^2+\int p_2^{\frac12}\psi_{r,\epsilon}|B^{12}|\left(\frac{\partial u}{\partial p_1}\right)^2\\
&\le Cr^{\frac12}\left(\int\psi_{r,\epsilon}p_2\left(\frac{\partial u}{\partial p_2}\right)^2+\int\psi_{r,\epsilon}\left(\frac{\partial u}{\partial p_1}\right)^2\right).
\end{aligned} \tag{4.28}$$

Combining the above estimates and choosing suitable r, δ, one can get

$$\int \psi_{r,\epsilon} p_2 \left(\frac{\partial u}{\partial p_2}\right)^2 + \int \psi_{r,\epsilon} \left(\frac{\partial u}{\partial p_1}\right)^2 \leq C \tag{4.29}$$

for some constant C independent of ϵ. Passing the limit $\epsilon \to 0$, we have finished the proof of the present lemma. $\square$

Now we can complete the proof of Case 2.2. Replacing $p_2 + \frac{y_2^k}{\mu_k^2}$ by p_2, still denote it by p_2. Then by Lemma 4.1, one can get $\bar{b}^2(p_1, 0) \geq b > 2$. All the coefficients of (4.14) are $C^1(\overline{R_+^2})$ with $\mu_k^{\frac{2\alpha}{\alpha-1}} \bar{f}(p, \mu_k^{-\frac{2}{\alpha-1}} v^k) \in L^\infty$, this means all the requirements in Lemma 4.2 are fulfilled. From Lemma 4.1, Lemma 4.2 and the regularity results of Theorem 5.1 and the standard regularity results for non-degenerate elliptic equations, one can choose a suitable subsequence such that $v^k(0', -\frac{y_2^k}{\mu_k^2}) \to v(0', c) \geq \frac{1}{2}$ and also $v^k \to v$ in the distribution sense in $\mathscr{D}'(R_+^2)$ and v satisfies

$$(p_2 + c)\partial_{y_2}(a^{ij}\phi_i\phi_j)(\bar{x})\frac{\partial^2 v}{\partial p_2^2} + a^{11}(\bar{x})\frac{\partial^2 v}{\partial p_1^2} + (b^i\phi_i + a^{ij}\partial_{ij}\phi)(\bar{x})\frac{\partial v}{\partial p_2} + h(\bar{x})v^\alpha = 0, \text{ in } R_+^2, \tag{4.30}$$

where $0 \leq c = \lim\limits_{k\to\infty} \frac{y_2^k}{\mu_k^2} < \infty$. By a linear change of coordinates and a stretching of coordinates, we have that

$$\begin{cases} p_2 v_{p_2p_2} + v_{p_1p_1} + \bar{b} v_{p_2} + v^\alpha = 0 \text{ in } R_+^2, \\ 0 \leq v \in C^2(R_+^2) \cap C(\overline{R_+^2}), v(0, c) = c_0 > 0. \end{cases} \tag{4.31}$$

From the assumption of Theorem 1.2, it follows that $2 < \bar{b} \leq a$ and

$$\alpha < \frac{2a+3}{2a-1} \leq \frac{2\bar{b}+3}{2\bar{b}-1}.$$

By Theorem 5.1, we see that $v \in C^2(\overline{R_+^2})$ and have $v \equiv 0$ which follows from Theorem 1.1. This is a contradiction to $v(0, c) > 0$. This ends the proof of Theorem 1.2.

5 Appendix

In the present Appendix, we shall give a result about the regularity of solutions to some degenerate elliptic equation in [14]. For the convenience of readers, we shall give a brief proof for it. We shall use the notations in [14]. Define $I_q(v)$ and $I_\beta(v)$ by:

$$I_q(v) = \|y\partial_{yy}v\|_{L^q(R_+^{n+1})} + \|\Lambda_1^2 v\|_{L^q(R_+^{n+1})} + \|y^{\frac{1}{2}}\Lambda_1 v_y\|_{L^q(R_+^{n+1})} + \|v_y\|_{L^q(R_+^{n+1})} + \|v\|_{L^q(R_+^{n+1})}, \tag{5.1}$$

$$I_\beta(v) = [y\partial_{yy}v]_{\dot{C}^\beta(\overline{R_+^{n+1}})} + [\Lambda_1^2 v]_{\dot{C}^\beta(\overline{R_+^{n+1}})} + [y^{\frac{1}{2}}\Lambda_1 v_y]_{\dot{C}^\beta(\overline{R_+^{n+1}})} + [v_y]_{\dot{C}^\beta(\overline{R_+^{n+1}})} + \|v\|_{L^\infty(R_+^{n+1})}, \tag{5.2}$$

where Λ_1 is a singular integral operator with the symbol $\sigma(\Lambda_1) = |\xi|$. Also we say a function $v(x, y)$ in $\dot{C}^\alpha(\overline{R_+^{n+1}})$, $\alpha \in R_+^1 \backslash Z$, if

$$|v|_{\dot{C}^\alpha(\overline{R_+^{n+1}})} = \sum_{|\beta| \leq [\alpha]} |D^\beta v|_{C(\overline{R_+^{n+1}})} + [v]_{\dot{C}^\alpha(\overline{R_+^{n+1}})} < \infty, \tag{5.3}$$

where

$$[v]_{\dot{C}^\alpha(\overline{R_+^{n+1}})} = \sum_{|\beta|=[\alpha]} \underset{y\geq 0, x\neq\bar{x}\in R^n}{sup} \Big(\frac{|D_x^\beta v(x,y) - D_x^\beta v(\bar{x}, y)|}{|x - \bar{x}|^\alpha}\Big). \tag{5.4}$$

Let $\psi \in C_c^\infty(R^{n+1})$ be a cutoff function with $\psi(x, y) = 1$ as $|x| \leq 1/2$, $y \leq 1/2$ and $\psi = 0$ as $|x| \geq 1$ or $|y| \geq 1$. Set $\psi_r(x, y) = \psi(\frac{x}{r}, \frac{y}{r})$.

Lemma 5.1. *(Lemma 5.4 in [14]) Suppose that $u \in C^2(R_+^{n+1}) \bigcap L^p(R_+^{n+1})$ with $u_x, yu_y \in L^p(R_+^{n+1})$ satisfies*

$$L(u) = yu_{yy} + \sum_{i,j} a_{ij}u_{x_ix_j} + y\sum_j a_j u_{yx_j} + \sum_j b_j u_{x_j} + bu_y = f, \text{ in } R_+^{n+1}, \tag{5.5}$$

where a_{ij}, a_j, b_j, b are all in $C(\overline{R_+^{n+1}})$ with $a_{ij}(0) = \delta_{ij}, b(0) > \frac{3}{2}$, $f \in L^\infty(R_+^{n+1})$ and that for some $\epsilon > 0$,

$$\lim_{y\to 0} y^{b(0)-1-\epsilon}u(x,y) = 0 \text{ uniformly for all } x \in R^n. \tag{5.6}$$

Then for sufficiently large p, there are $r = r(p) > 0$ such that

$$I_p(\psi_r u) \le C_r, \tag{5.7}$$

for some constant depending only on $p, \|\psi_{2r}f\|_{L^p}, \|\psi_{2r}u\|_{L^p}, \|\psi_{2r}u_x\|_{L^p}$ and $\|y\psi_{2r}u_y\|_{L^p}$ provided that $p > n+1$ or $p > \frac{n+1}{2}$ and $b(0) - 2 - \epsilon > 0$.

Lemma 5.2. *(Lemma 5.5 in [14]) Suppose that $w, \partial_x w, y\partial_y w \in \dot{C}^\alpha_{loc}(\overline{R_+^{n+1}}) \cap C^2(R_+^{n+1})$ with $\alpha \in R_+^1 \backslash Z$ and w satisfies (5.5),where a_{ij}, a_j, b_j, b, f are all in $\dot{C}^\alpha_{loc}(\overline{R_+^{n+1}})$ with $a_{ij}(0) = \delta_{ij}, b(0) > \frac{3}{2}$. Then*

$$I_\alpha(\psi_r w) \le C, \tag{5.8}$$

for some positive constants r and C, depending on $\alpha, |\psi_{2r}f|_\alpha, |\psi_{2r}w|_\alpha$, $|\psi_{2r}\partial_x w|_\alpha$ and $|y\psi_{2r}\partial_y w|_\alpha$.

Denote by $W^{1,p}_\alpha(U)$ the completion of the space of all the functions u in $C^1(\bar{U})$ under the norm

$$\Big(\int_U y^{p\alpha}|Du|^p dxdy + \int_U y^{p\alpha}|u|^p dxdy\Big)^{\frac{1}{p}}.$$

Here we always assume $U \subset R_+^{n+1}$, bounded and $\partial U \cap \{y = 0\}$ nonempty.

Lemma 5.3. *(Lemma 8.3 in [14] Appendix B) Let $U \in C^1$ be bounded domain and let $\alpha \in (0,1)$. Then the following maps are continuous*

$$W^{1,p}_\alpha(U) \hookrightarrow C^\beta(\bar{U}) \text{ where } \beta = 1 - \alpha - \frac{n+1}{p}, \text{ if } p > \frac{n+1}{(1-\alpha)}, \tag{5.9}$$

$$W^{1,p}_\alpha(U) \hookrightarrow L^q(U) \text{ where } q < \frac{(n+1)p}{n+1-(1-\alpha)p}, \text{ if } \frac{1}{1-\alpha} < p < \frac{n+1}{(1-\alpha)}. \tag{5.10}$$

Moreover, for $p = 2$ and $\forall \alpha \in (0,1)$, one can have

$$W^{1,2}_\alpha(U) \hookrightarrow L^q(U) \text{ where } q < \frac{q_1}{1+2\alpha} \text{ and } q_1 = 2 + \frac{4}{n}. \tag{5.11}$$

With the above three lemmas, we can establish the following theorem concerning the regularity of solutions to degenerate elliptic equation (5.5) for $n = 1$ and $p = 2$.

Theorem 5.1. *Suppose that $b(0) > 2$ and $u \in C^2(R_+^2) \bigcap L^\infty(R_+^2)$ with $u_x, yu_y \in L^2(R_+^2)$ satisfies* (5.5). *Then*

(1) Suppose $a_{ij}, a_j, b_j, b \in C(\overline{R_+^2})$ and $f \in L^\infty(R_+^2)$. Then there exist two constants $r > 0$ and $\beta \in (0,1)$ such that

$$\|\psi_r u\|_{C^\beta(\overline{R_+^2})} + \|\psi_r u_x\|_{C^\beta(\overline{R_+^2})} + \|y\psi_r u_y\|_{C^\beta(\overline{R_+^2})} \le C_r, \tag{5.12}$$

for some constant depending only on $\|\psi_{2r}f\|_{L^\infty}, \|\psi_{2r}u\|_{L^\infty}, \|\psi_{2r}u_x\|_{L^2}$ and $\|y\psi_{2r}u_y\|_{L^2}$.

(2) *Suppose* $a_{ij}, a_j, b_j, b, f \in \dot{C}^{k+\beta}(\overline{R^2_+}), k \geq 0$. *Then there exist two constants* $r > 0$ *and* $\beta \in (0,1)$ *such that*

$$I_{k+\beta}(\psi_r u) \leq C_r, \tag{5.13}$$

for some constant depending only on $\|\psi_{2r} f\|_{C^k}, \|\psi_{2r} u\|_{L^\infty}, \|\psi_{2r} u_x\|_{L^2}$, $\|y\psi_{2r} u_y\|_{L^2}$ *and the* C^k*-norm of the coefficients.*

Proof. We first prove (5.12). By Lemma 5.1, one can get

$$I_2(\psi_r u) \leq C_r, i.e., y\psi_r u \in H^2(R^2_+), y^{\frac{1}{2}}\psi_r u_{xy} \in L^2(R^2_+).$$

Hence by Sobolev embedding theorem, it follows that $y\psi_r u_y \in L^p(R^2_+), \forall p \in [2, \infty)$. By noting $\psi_r u_x \in W^{1,2}_{\frac{1}{2}}(R^2_+)$ and (5.11)(where $n = 1$), we can see that $\psi_r u_x \in L^{p_1}(R^2_+), \forall p_1 \in [2, 3)$. Now we can apply Lemma 5.1 again for p_1 and another smaller r_1(for simplicity we always denote it by r) to get

$$I_{p_1}(\psi_r u) \leq C_r, i.e., \psi_r u_x \in W^{1,p_1}_{\frac{1}{2}}(R^2_+), \psi_r u_x, \psi_r u_y \in L^{p_1}(R^2_+).$$

This implies that $\psi_r u \in W^{1,p_1}(R^2_+)$. Then $\psi_r u \in C^{\frac{1}{5}}(\overline{R^2_+})$ if we take $p_1 = \frac{5}{2}$. Using Lemma 5.3 again, one can get $\psi_r u_x \in L^{p_2}(R^2_+), \forall p_2 \in [2, 12)$. Again, by Lemma 5.1, one can get

$$I_{p_2}(\psi_r u) \leq C_r, i.e., \psi_r u_x \in W^{1,p_2}_{\frac{1}{2}}(R^2_+).$$

By Lemma 5.3, we can get

$$\|\psi_{r_2} u\|_{C^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r u_x\|_{C^{\frac{1}{5}}(\overline{R^2_+})} + \|y\psi_r u_y\|_{C^{\frac{1}{5}}(\overline{R^2_+})} \leq C_r, \text{ if } p_2 = \frac{20}{3}. \tag{5.14}$$

This proves (5.12). Now we can prove (5.13) by induction on k. For $k = 0$, (5.14) means we can apply Lemma 5.2 to get

$$\|\psi_r y u_{yy}\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r u_{xx}\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|y^{\frac{1}{2}}\psi_r u_{xy}\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r u_y\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} \leq C_r$$

For $k = 1$, as $\psi_r \partial_x u_x, y^{\frac{1}{2}}\psi_r \partial_y u_x \in C^{\frac{1}{5}}(\overline{R^2_+})$, we can continue to apply Lemma 5.2 to $\psi_r u_x$ again to get $I_{\frac{1}{5}}(\psi_r u_x) \leq C_r$ namely,

$$\|\psi_r y \partial_{yy}(u_x)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r \partial_{xx}(u_x)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r \partial_y(u_x)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} \leq C_r$$

This means $\psi_r \partial_x(u_y) \in \dot{C}^{\frac{1}{5}}(\overline{R^2_+})$. Combining with $\psi_r y \partial_y(u_y) \in \dot{C}^{\frac{1}{5}}(\overline{R^2_+})$ and applying Lemma 5.2 to $\psi_r u_y$ again, we can see that $I_{\frac{1}{5}}(\psi_r u_y) \leq C_r$, namely,

$$\|\psi_r y \partial_{yy}(u_y)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r \partial_{xx}(u_y)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} + \|\psi_r \partial_y(u_y)\|_{\dot{C}^{\frac{1}{5}}(\overline{R^2_+})} \leq C_r$$

Also this implies that $u_{yy} \in C(\overline{R^2_+})$. For general k, repeat the above steps, we can get (5.13). □

Acknowledgement

This work is supported by a training program for innovative talents of key disciplines, Fudan University. The author would like to thank the valuable suggestions of Professor J.X.Hong and Professor C.M.Li.

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